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AUTHOR	Brainerd, Charles J.
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ABSTRACT

The purpose of this paper is to provide a formal three-step explication of a question that has been traditionally viewed as fundamental to developmental theory construction and explanation: Is behavioral development continous or discontinuous, or is it both? First, a preliminary structural definition of behavioral development involving a certain domain of elementary phenomena ("behavioral events in living systems") and a certain elementary relation ("precedes in time") is presented and rationalized. Second, formal models derived from the continuous connected straight line of classical mathematics are constructed for each of the following assertions: (1) Behavioral development is continuous. (2) Behavioral development is discontinuous. (3) Behavioral development is partially discontinuous. Third, the differences between the respective formal models are summarized. It is shown that each of the three models can be applied to any narrowly defined behavioral content area whatsoever and that, taken together, the models entail a new empirical approach to the old question of whether or not behavioral development in this or that content area is continuous. (Author/CS)

FORMAL MODELS OF CONTINUITY, DISCONTINUITY, AND PARTIAL DISCONTINUITY FOR BEHAVIORAL DEVELOPMENT

Charles J. Brainerd

Center for Advanced Study in Theoretical Psychology--University of Alberta

The meaning of the question, "Is behavioral development continous or discontinuous or is it both?" is formally explicated in three steps. First, a preliminary structural definition of behavioral development involving a certain domain of elementary phenomena ("behavioral events in living systems") and a certain elementary relation ("precedes in time") is presented and rationalized. Second, formal models derived from the continuous connected straight line of classical mathematics are constructed for each of the following assertions: (1) Behavioral development is continuous. (2) Behavioral development is discontinuous. (3) Behavioral development is partially discontinuous. Third, the differences between the respective formal models are summarized. It is shown that each of the three models can be applied to any narrowly defined behavioral content area whatsoever and that, taken together, the models entail a new empirical approach to the old question of whether or not behavioral development in this or that content area is continuous.

FORMAL MODELS OF CONTINUITY, DISCONTINUITY, AND PARTIAL DISCONTINUITY FOR BEHAVIORAL DEVELOPMENT

Charles J. Brainerd

Center for Advanced Study in Theoretical Psychology--University of Alberta

The purpose of this paper is to provide a purely formal explication of a question that traditionally has been viewed as fundamental to developmental theory construction and explanation: Is behavioral development continuous or discontinuous or is it both? To avoid repetition, this will be termed "the continuity question" hereafter. Historically, this question has been one of the most disputed issues in the foundations of developmental psychology. For example AUSUBEL and SULLIVAN recently have observed that "Second only to the nature-nurture controversy has been the great debate over whether development in [sic] a process of gradual quantitative and continuous change, or whether it is characterized by abrupt, uneven and discontinuous changes which are qualitatively different from one another [1970, p. 98]."

Although the question of whether or not specific domains of phenomena are continuous has been examined with definite profit in mathematics, the physical sciences, the biological sciences, a even some areas of psychology [e.g., BOWER and TRABASSO, 1963; WERTHEIMER, 1972], the continuity question has proved singularly intractable with reference to behavioral development. This causes one to wonder why developmental psychology should be an exception to the rule. The most prevalent current opinion seems to be that, for whatever reasons, the continuity question becomes inherently intractable when it is applied to behavioral development [e.g., cf. AUSUBEL and SULLIVAN, 1970]. There is another explanation however. It seems very probable that the intractability of the continuity question stems from the veritable plethora of meanings that have been assigned to the terms "continuous" and "discontinuous" by developmental psychologists. Here, I am referring to the meanings that have been assigned to these terms vis-à-vis behavioral development in general, rather than meanings that have been employed with reference to some narrowly defined behavioral content area [e.g., KAGAN, 1971]. While these terms have been assigned rather precise meanings in mathematics and the physical sciences, vague and sometimes contradictory definitions characterize the developmental literature. To illustrate, consider WERNER'S [1957] and AUSUBEL and SULLIVAN'S [1970] repective definitions. WERNER specifically excludes from his definitions of "continuous" and "discontinuous" certain notions, such as "quantitative" and "qualitative," which AUSUBEL and SULLIVAN incorporate as essential components of their definitions. PIAGET'S definitions of the same two concepts [e.g., PIAGET, 1956, 1960] constitute yet another distinct position in which obscure notions such as "equilibration" and "structural change" are viewed as essential.

In short, there seems to be ample evidence that an explication of the meanings of "continuous" and discontinuous" as they apply to behavioral development in general is very much in order. Rather than attempt to review and analyze all that has been said about these notions in the developmental literature, I shall employ a purely formal mode of explication. To my mind, the principal virture of such an approach is that it allows us to establish the basic points of difference and agreement without the incursion of extraneous meanings. The explication proceeds in three steps. First, behavioral development is tentatively defined in terms of two primitive concepts which it presupposes. Because the two defining concepts are viewed as primitive herein, they are not themselves defined in any strict sense. However, certain descriptive assumptions are made about each and a rationale for choosing them as defining attributes of the concept of behavioral development is presented. Second, three formal models are proposed which define continuity, discontinuity, and partial discontinuity with reference to the continuous connected straight line. A developmental interpretation of each model is generated simply by assigning the primitive concepts that behavioral development presupposes as interpretations of the variables of each model. Once the variables are so interpreted, each model becomes an axiomatic definition of one of the following statements: (1) Behavioral development is continuous. (2) Behavioral development is discontinuous. (3) Behavioral development is partially discontinuous. Third, the key differences between the models are summarized and possible new approaches to the empirical study of the continuity question are discussed.

Before proceeding, it should be noted that the writer takes no a priori position concerning which of the three formal models to be proposed provides the best characterization of behavioral development. Although there can be little doubt that the a priori preferences of most working developmental psychologists would lie with the continuity model [e.g., cf. KESSEN, 1962], the writer views the validation of the respective models in specific behavioral content areas as purely empirical question.

Preliminary Definition of Behavioral Development

As is always the case with formal analyses of scientific constructs, we must begin with a definition of the domain of interest in terms of certain undefined notions. This is necessitated by the fact that models of the type we shall consider establish relationships between uninterpreted symbols, rather than between specific classes of referent phenomena. A model of this type can be said to be a model of some given domain of interest (e.g., behavioral development in our case) only if the symbols have been assigned the undefined notions of that domain as interpretations. Hence, our first task must be to arrive at some intuitively reasonable formulation of the rudimentary ideas that "behavioral development" presupposes.

The Definition

I shall characterize behavioral development in terms of a certain domain of elementary (for psychology) phenomena and a certain relation on that domain. Consider a nonempty set D whose members are symbolized $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$, ... and a relation T on D. If each of the $[\underline{x}_1, \underline{x}_2, \underline{x}_3 \ldots] \in D$ is assigned the interpretation "a behavioral event in a living system" and the relation T on D is assigned the interpretation "precedes in time," then we shall say that the system [D, T] comprises a formal definition of behavioral development. Under these interpretations, all expressions of the form $\underline{x}_i T \underline{x}_j$ are read: The behavioral event $\underline{x}_i$ precedes the behavioral event $\underline{x}_j$ in time, where it is understood that $(\underline{x}_i, \underline{x}_j) \in D$. D and T are the only undefined notions that we shall require to interpret our three models and to explicate the terms "continuous" and "discontinuous" as they apply to behavioral development.

Although behavioral events in living systems must be left (strictly speaking) undefined, two further assumptions are made about these elementary phenomena. First, in view of the fact that developmental psychology purports to be an empirical rather than a rational discipline, we shall stipulate that the $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$, ... are at least potentially measurable. Second, we shall stipulate that our knowledge of the meaning of "a behavioral event in a living system," while quite obviously informal, is sufficiently precise to allow us to employ the notion without blatant inconsistency.

To avoid subsequent repetition, we shall stipulate here that the relation T is <u>a simple ordering relation</u> and that the system [D, T] is <u>completely unbounded</u>. Concerning the former point, to say that T is a simple ordering relation is to say that it is both asymmetrical and transitive. That is,

(1) for any nonidentical $(x_{\underline{i}}, x_{\underline{j}}) \in D$, either $\underline{x}_{\underline{i}} T \underline{x}_{\underline{j}}$ or $\underline{x}_{\underline{j}} T \underline{x}_{\underline{i}}$ but not both [asymmetry], and

(2) for any nonidentical $(\underline{x}_{\underline{i}}, \underline{x}_{\underline{j}}, \underline{x}_{\underline{k}}) \in D$, if $\underline{x}_{\underline{i}} T \underline{x}_{\underline{j}}$ and $\underline{x}_{\underline{j}} T \underline{x}_{\underline{k}}$, then $x_i T x_k$ also [transitivty].

Concerning the latter point, to say that [D, T] is completely unbounded is to say that D is not known to have either a "first" element or a "last" element. That is,

(3) there is no $\underline{x}_{\underline{i}} \in D$ such that $\underline{x}_{\underline{i}} T \underline{x}_{\underline{j}}$ for every other nonidentical $\underline{x}_{\underline{j}} \in D$ [lower unboundedness], and

(4) there is no $\underline{x}_k \in D$ such that $\underline{x}_j T \underline{x}_k$ for every other nonidentical $\underline{x}_j \in D$ [upper unboundedness].

Rationale

The undefined notions. I do not believe that there can be any serious objection to employing the relation "precedes in time" as a defining attribute of behavioral development. While it must be acknowledged that no agreement exists on the fine details of how the concept of development is to be defined vis-à-vis behavior [e.g., compare ANDERSON, 1957; NAGEL, 1957; WERNER, 1957], there is extensive agreement on the following point: Whatever else it may entail with regard to specific domains of application (biology, psychology, etc.), development always denotes a time ordered process in a living system [HARRIS, 1957; WOHLWILL, 1970].

There appear to be two prima facie objections that can be lodged against using "a behavioral event in a living system" as a defining attribute of behavioral development. First, although there probably would be few objections to treating behavioral events as elementary phenomena in most other branches of psychology, developmental psychologists traditionally have thought of themselves as studying behavioral changes within and between generations. Hence, it might reasonably be argued that each of the $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$, ... should be assigned the interpretation "a behavioral change in a living system." Although this interpretation seems intuitively appropriate, it can be vitiated on both logical and empirical grounds. Logically, the notion "behavioral change" is not elementary. In fact, it is compounded from and can be reduced to (i.e., satisfactorily defined in terms of) the domain D and the relation T under their stated interpretations. Behavioral changes, at least those studied by developmental psychologists, are inferred differences between behavioral events measured at different times. The most rudimentary example of a behavioral change would be an inferred difference between two elementary behavioral events (perceptual discriminations scores on a personality test, etc.) measured at different times. Hence, we may define the concept "behavioral change" as that special subset of the set of all expressions of the form $\underline{x}_i T \underline{x}_j$ whose members satisfy the additional requirement that $\underline{x}_i \neq \underline{x}_j$. Empirically, it usually is deemed essential that the elementary phenomena of any scientific discipline be directly measured. Although behavioral events in living systems are satisfactory in this respect, behavioral changes quite obviously are not.

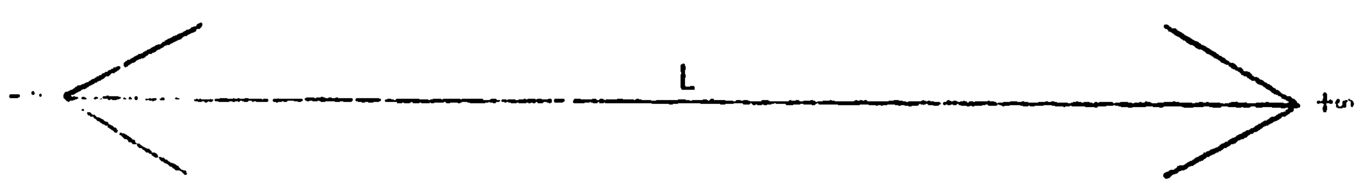
The second objection to treating behavioral events in living systems as the elementary phenomena of behavioral development is that if we accept the testimony of the biological sciences, behavioral events are neither irreducible nor unanalyzable. Quite to the contrary, they appear to be reducible to more basic physiological and experiential events. The objection, then, is this: If we admit at the outset that behavioral events are not irreducible in some ultimate sense, then is it not logically inappropriate to treat them as elementary phenomena in <u>any</u> definition? It turns out that this usage is not logically flawed because it is simply incorrect to suppose that the inclusion of a certain class of occurrences as elementary phenomena is a given definition somehow entails that these occurrences are irreducible in some ultimate sense [cf. RUSSELL, 1948]. Instead, it is possible to view certain phenomena as elementary for purposes of some specific formal definition, while simultaneously acknowledging that they are characterized by a complex structure. Regardless of whether the elementary phenomena of a formal definition are or are not ultimately irreducible, the expressions that result from combining the symbols that represent these phenomena in formal models will be exactly the same. Therefore, we may treat behavioral events as elementary phenomena for the sake of defining behavioral development and simultaneously admit that there are more basic levels of analysis. For those who find this situation somewhat counterintuitive, it should be noted that a quite analogous situation has existed for some time in theoretical physics. On the one hand, space-time loci are viewed as elementary for purposes of formally defining the concept "matter" while, on the other hand, space-time loci are definable for other purposes [RUSSELL, 1928, 1948].

Properties of the undefined notions. As was the case for the relation "precedes in time" itself, there appear to be no serious objections that can be lodged against the assumption that this is a simple ordering relation. Our common sense conception of this relation implies both asymmetry and transitivity: For every pair of distinct occurrences A and B separated by some temporal interval, if we are given that A occurs before B, then we invariably conclude that the reverse is not true; for every triplet of distinct events A, B, and C, if we are given that A occurs before B and B occurs before C, then we invariably conclude that A occurs before C. In addition to our common sense conception, formal theories of temporal relations (e.g., those of physics) routinely assign the property of simple ordering to time [RUSSELL, 1903].

The complete unboundedness of [D, T] under the stated interpretations is less obvious than the simple ordering of T. In fact, it can be argued that the system has both an upper and a lower bound. This argument turns on the assumption that behavioral development in any organism has a definite beginning (conception) and the assumption that it has a definite end (death). Although these objections may seem well-taken, both reflect ontogenetic myopia. That is, they implicitly assume that behavioral development is synonymous with ontogenesis. This assumption is unwarranted. It is widely conceded that the concept of behavioral development must include both ontogenesis and phylogenesis [HARRIS, 1957]. Both lower and upper boundedness are specifically proscribed in modern evolutionary theory for the following reasons. Lower boundedness may be taken both to imply special creation of some sort and to imply that a sharp line can be drawn between living and nonliving organic compounds. Upper boundedness entails that evolution either has ceased or will cease at some future time.

Three Formal Models

We shall consider formal models of continuity, discontinuity, and partial discontinuity in this section. Before proceeding, however, some historical remarks about the concept of continuity are in order. Since the time of Pythagoras and his brotherhood, the continuous connected straight line L has been accepted as the embodiment of the concept of continuity [RUSSELL, 1903].



That is, in both mathematics and the empirical sciences, the assertion that this or that system is continuous traditionally has been taken to mean that L is a representation of that system (i.e., a one-to-one correspondence can be established between the elements of that system and the points of L). Since the discovery of higher-order continua, such as those that characterize the complex numbers and the space-time manifold of relativity theory, L has also been called the first-order continuum in in recognition of the fact that other continua ultimately reduce to L.

As was the case for behavioral development, L may be formally defined in terms of a certain domain of elements and a certain relation on that domain. Consider a nonempty set S whose members are symbolized s_1, s_2, s_3 ... and a relation P on S. If each of the $[s_1, s_2, s_3, \ldots] \epsilon S$ is assigned the interpretation "a point on a plane" and the relation P on S is assigned the interpretation "to the left of," then the system [S, P] is a formal definition of L. The property of simple ordering is stipulated for P and the property of complete unboundedness is stipulated for [S, P].

Now, let us consider what it means to have a formal model of continuity. Consider some completely unbounded set of elements S' and a simple ordering relation P' on S'. Suppose that there is some function f that maps each and every element of S' with a unique element of S such that the relation P' is preserved the relation P. We shall denote this mapping f: $S' \rightarrow S$. Because L is our informal model of continuity, we can express continuity, discontinuity, and partial discontinuity formally in terms of f $S' \rightarrow S$ as follows.

(1) Continuity. Between all pairs of nonidentical points of L that are mapped with elements of S', there are no points of L that are not mapped with elements of S'. In other words, for all pairs of nonidentical $(s_i, s_j) \epsilon S$ such that $s_i P s_j$ and for all pairs of nonidentical $(s_i', s_j') \epsilon S'$

such that $\underline{f}$: $\underline{s}_i{}' \rightarrow \underline{s}_i$ and $\underline{f}$: $\underline{s}_j{}' \rightarrow \underline{s}_j$ for every $\underline{s}_k \varepsilon S$ such that $\underline{s}_i P \underline{s}_k$ and $\underline{s}_k P \underline{s}_j$ there is some $\underline{s}_k{}' \varepsilon S'$ such that $\underline{f}$: $\underline{s}_k{}' \rightarrow \underline{s}_k$. In short, the mapping of [S', P'] onto L leaves no gaps on L.

(2) Discontinuity. Between all pairs of nonidentical points of L that are mapped with elements of S', there is at least one point of L that is not mapped with any element of S'. In other words, for all pairs of nonidentical $(\underline{s}_i, \underline{s}_j) \varepsilon S$ such that $\underline{s}_i P \underline{s}_j$ and for all pairs of nonidentical $(\underline{s}_i{}', \underline{s}_j{}') \varepsilon S'$ such that $\underline{f}$: $\underline{s}_i{}' \rightarrow \underline{s}_i$ and $\underline{f}$: $\underline{s}_j{}' \rightarrow \underline{s}_j$, for some $\underline{s}_k \varepsilon S$ such that $\underline{s}_i P \underline{s}_k$ and $\underline{s}_k P \underline{s}_j$ there is no $\underline{s}_k{}' \varepsilon S$ such that $\underline{f}$: $\underline{s}_k{}' \rightarrow \underline{s}_k$. In short, the mapping of [S', P'] onto L leaves gaps on L between every pair of nonidentical points on L.

(3) Partial discontinuity. Between some pairs of nonidentical points on L that are mapped with elements of S', there is at least one point of L that is not mapped with any element of S'. In other words, for some pairs of nonidentical $(\underline{s}_i, \underline{s}_j) \varepsilon S$ such that $\underline{s}_i P \underline{s}_j$ and for some pairs of nonidentical $(\underline{s}_i{}', \underline{s}_j{}') \varepsilon S'$ such that $\underline{f}$: $\underline{s}_i{}' \rightarrow \underline{s}_i$ and $\underline{f}$: $\underline{s}_j{}' \rightarrow \underline{s}_j$, for some $\underline{s}_k \varepsilon S$ such that $\underline{s}_i P \underline{s}_k$ and $\underline{s}_k P \underline{s}_j$ there is no $\underline{s}_k{}' \varepsilon S'$ such that $\underline{f}$: $\underline{s}_k{}' \rightarrow \underline{s}_k$. In short, the mapping of [S, P'] onto L leaves gaps on L between some pairs of nonidentical points on L.

Formal analyses of L conducted during the nineteenth century by CANTOR [cf. RUSSELL, 1903, chapters XXXV and XXXVI] and DEDEKIND [1901] indicated that for the system [S', P'] to be a model of L (i.e., for $\underline{f}$: $S' \rightarrow S$ to leave no gaps on L), the system must satisfy six conditions: (1) asymmetry, (2) transitivity, (3) upper unboundedness, (4) lower unboundedness, (5) density, and (6) completeness. Properties 1 through

4 already have been considered in conjunction with our formal definition of behavioral development. Concerning property 5, S' is said to be "dense" if the following condition is satisfied: For every pair of nonidentical $(\underline{s_i}', \underline{s_j}') \in S'$ such that $\underline{s_i}'P'\underline{s_j}'$, there is some $\underline{s_k}' \in S'$ such that $\underline{s_i}'P'\underline{s_k}'$ and $\underline{s_k}'P'\underline{s_j}'$. Thus, the density property stipulates that "between" every pair of nonidentical elements of S' there is at least one other nonidentical element. Concerning property 6, suppose that we partition [S', P'] into two subsystems [M', P'] and [N', P'] such that $\underline{s_i}'P'\underline{s_j}'$ for every $\underline{s_i}' \in M'$ and $\underline{s_j}' \in N'$. Hereafter, any such partitioning of a completely unbounded system composed of a nonempty set and a simple ordering relation on that set will be termed an <u>ordinal partition</u>. The system [S', P'] is said to be complete if the following condition is satisfied: For every ordinal partition of S', if there is no $\underline{s_m}' \in M'$ such that $\underline{s_i}'P'\underline{s_m}'$ for every other nonidentical $\underline{s_i}' \in M'$, then there may or may not be some $\underline{s_n}' \in N'$ such that $\underline{s_n}'P'\underline{s_j}'$ for every other nonidentical $\underline{s_j}' \in N'$. A system is called "incomplete" if there is always some $\underline{s_n}' \in N'$ such that $\underline{s_n}'P'\underline{s_j}'$ for every other nonidentical $\underline{s_j}' \in N'$ whenever there is no $\underline{s_m}' \in M'$ such that $\underline{s_i}'P'\underline{s_j}'$. In essence, the completeness property stipulates that when the "lower" segment of an ordinal partition of any continuous system contains no "last" element, the "upper" segment of that partition does not necessarily contain a "first" element.[1] Conditions 1 through 5 normally are called the necessary conditions continuity. Condition 6 normally is called the sufficient condition for continuity; any system that satisfies this condition is continuous.

Model I: Continuity

Statement of the model. In this model, it is stipulated that all six conditions for continuity hold for the system [S', P']. If we map each and every element of S' with one and only one point of L in such a manner that the relation P' on S' is preserved by the relation P on L, then the outcome of this mapping must be the first of the three outcomes described above. To illustrate, suppose that there was some $s_k \varepsilon S$ for which there was no $s_k' \varepsilon S'$ such that f: $s_k' \rightarrow s_k$. Logically, there are only two situations which could produce this result. First, there is some pair of nonidentical $(s_i', s_j') \varepsilon S'$ such that f: $s_i' \rightarrow s_i$ and f: $s_j' \rightarrow s_j$ for which there is no s_k' such that $s_i'P's_k'$ and $s_k'P's_j'$ but there is some $s_k \varepsilon S$ such that $s_i P s_k$ and $s_k P s_j$. In other words, there is at least one point on L "between" the two points with which s_i' and s_j' are mapped but there is no corresponding point between s_i' and s_j'. Second, for any ordinal partitioning of [S', P'] into the nonempty subsets M' and N', whenever there is no $s_m' \varepsilon M'$ such that $s_i'P's_m'$ for all other nonidentical $s_i' \varepsilon M'$, there is always some $s_n' \varepsilon N'$ such that $s_n'P's_j'$ for all other nonidentical $s_j' \varepsilon N'$. That is, N' always has a "first" element whenever M' does not have a "last" element. However, we know from condition 5 above that the first situation cannot arise in conjunction with [S', P'] and we know from condition 6 above that second situation cannot arise either. Hence, the mapping of the elements of S' with the points of L must yield a one-to-one correspondence and, therefore, the system [S', P'] is continuous.

Developmental representation of Model I. Suppose that we interpret

S' as D and P' as T. That is, each of the elements of our abstract domain are assigned the interpretation "a behavioral event in a living system" and our abstract simple ordering relation is assigned the interpretation "precedes in time." Because it already has been assumed that the properties of simple ordering and unboundedness hold for [D, T] independent of its continuity or discontinuity, it is the developmental representations of conditions 5 and 6 that are the crucial aspects of this first model. By virtue of condition 5, we stipulate that [D, T] is dense. By virtue of condition 6, we stipulate that [D, T] also is complete. Hence, if each and every behavioral event of D is mapped with one and only one point along L in such a manner that the relation "precedes in time" is preserved by the relation "to the left of", the mapping would leave no gaps on L.

Numerical representation of the model. The system of real numbers is the appropriate numerical representation of Model I. That is, if S' is interpreted as the set of all real numbers and P' is interpreted as the relation "less than," then the system [S', P'] is a model of the real numbers (or, alternatively, the system of real numbers satisfies the six conditions of Model I). For present purposes we shall define the real numbers somewhat informally as the union of the set of all rational numbers and the set of all irrational numbers--where a rational number is any number of the form a/n (where n is any number from the set 1, 2, 3, ... and a is any number from the set 0, ±1, ±2, ±3 ...) and an irrational number is a number that cannot be expressed as a quotient of the form a/n (e.g., $\sqrt{2}$ or $\sqrt{3}$). It is well-known that the axioms of the real number system are the six properties of the continuous connected straight line L discussed earlier [e.g., cf. BEAUMONT and PIERCE, 1963; FEFERMAN, 1964; NIVEN, 1961]. That is, the real number axioms are our six conditions for continuity under numerical interpretations. In fact, the real numbers usually are considered the textbook illustration of a model of L.

Model II: Discontinuity

Statement of the model. First, it is stipulated that conditions 1 through 4 for continuity hold for [S', P'] but that conditions 5 and 6 do not. Contrary to condition 5, it is stipulated that for every $s_i' \in S'$ there is some $s_j' \in S'$ for which there is no $s_k' \in S'$ such that $s_i'P's_k'$ and $s_k'P's_j'$. In view of the fact that condition 5 is a necessary prerequisite for condition 6, condition 6 does not hold for [S', P'] either. (If the system does not satisfy condition 5, then obviously there can be no ordinal partition of the system in which the "lower" subset does not have a "last" element.)

Suppose that we now map each and every element of S' with one and only one point of L in such a manner that the relation P' on S' is preserved by the relation P on L. An informal proof shows that the result of the mapping f: $S' \to S$ must be the second of the three outcomes described above. Consider some arbitrary $s_i' \in S'$ and some arbitrary $s_i \in S$ such that f: $s_i' \to s_i$.

(1) By our ammendment of condition 5, there is some $s_j' \in S'$ such that if $s_i'P's_j'$ then there is no $s_k' \in S'$ such that both $s_i'P's_k'$ and $s_k'P's_j'$.

(3) Because condition 5 holds for the system [S, P], there is some $\underline{s}_k \varepsilon S$ such that both $\underline{s}_i P \underline{s}_k$ and $\underline{s}_k P \underline{s}_j$. By statement 1, however, there is no $s_k' \varepsilon S'$ such that both $\underline{s}_i'P'\underline{s}_k'$ and $\underline{s}_k'P'\underline{s}_j'$.

(4) By $\underline{f}$: $\underline{s}_i' \to \underline{s}_i$ and $\underline{f}$: $\underline{s}_j' \to \underline{s}_j$, then, there is no $\underline{s}_k' \varepsilon S'$ such that $\underline{f}$: $\underline{s}_k' \to \underline{s}_k$.

(5) Because both $\underline{s}_i'$ and $\underline{s}_i$ were arbitrarily chosen, it follows that $\underline{f}$: $S' \to S$ leaves a gap on L between any two nonidentical points ($\underline{s}_i$, $\underline{s}_j$) such that $\underline{s}_i P \underline{s}_j$.

Developmental representation of Model II. Suppose that we again interpret S' as D and P' as T. By earlier assumptions, the properties of simple ordering and unboundedness hold under these interpretations. By the above ammendment of condition 5, the system [D, T] is not dense. Hence, it also is not complete. Under these interpretations the preceding informal proof establishes that if we were to map each and every behavioral event in D with one and only one point along L in such a manner that "precedes in time" is preserved by "to the left of," this mapping would leave a gap on L between every pair of nonidentical points ($\underline{s}_i$, $\underline{s}_j$) such that $\underline{s}_i$ is to the left of $\underline{s}_j$.

Numerical representation of Model II. The appropriate numerical representation of Model II is the series 0, ±1, ±2, ±3, ... of integers. That is, if S' is interpreted as the set of all integers and P' is interpreted as the relation "less than," then the system [S', P'] is a model of the series of integers. As was the case for the real number representation of Model I, the properties of the series of integers are well-known. These properties are simple ordering, complete unboundedness, and our ammended version of condition 5 [e.g., cf. BEAUMONT and PIERCE, 1963; FEFERMAN, 1964; NIVEN, 1961]. In other words, the integer axioms are our conditions for discontinuity. Hence, if each and every integer is mapped with one and only one point along L in such a manner that "less than" is preserved by "to the left of," the mapping would leave a gap on L between every pair of nonidentical points ($\underline{s}_i$, $\underline{s}_j$) such that $\underline{s}_i$ is to the left of $\underline{s}_j$.

Model III: Partial Discontinuity

Statement of the model. It is stipulated that all the necessary conditions for continuity (i.e., 1 through 5) hold for the system [S', P'] but that the sufficient condition for continuity does not hold. The sixth condition is ammended as follows: For any ordinal partitioning of S' into the nonempty subsets M' and N' such that $\underline{s}_i$'P'$\underline{s}_j$' for every $\underline{s}_i' \in M'$ and every $\underline{s}_j' \in N'$, if there is no $\underline{s}_m' \in M'$ such that $\underline{s}_i$'P'$\underline{s}_m$' for every other nonidentical $\underline{s}_i' \in M'$ then there is always some $\underline{s}_n' \in N'$ such that $\underline{s}_n$'P'$\underline{s}_j$' for every other nonidentical $\underline{s}_j' \in N'$. Thus, whenever the "lower" subset of the partition contains no "last" element, the "upper" subset must contain a "first" element. Suppose that we again map each and every element of S' with one and only one point along L in such a manner that the relation P' on S' is preserved by the relation P on L. Another informal proof shows that the result of $\underline{f}$: S' → S must be the third of the three outcomes described earlier:

Suppose that S' is ordinally partitioned such that the nonempty subsets M' and N' are, respectively, the "lower" and "upper" subsets of the partition. Suppose also that M' has no "last" element. Finally, suppose that M is the nonempty subset of S onto which f: $S' \to S$ maps the elements of M' and that N is the nonempty subset of S onto which f: $S' \to S$ maps the elements of N'.

(1) By our amendment of condition 6, there is some $s_n' \in N'$ such that $s_n'P's_j'$ for every other nonidentical $s_j' \in N'$. Hence, there is no $s_j' \in S'$ such that $s_j'P's_k'$.

(2) Because f: $S' \to S$ preserves the relation P' on S', the nonempty subset N must be the "upper" subset of some ordinal partition of S. By condition 6, there may or may not be some $s_n \in S$ such that s_nPs_j for every other nonidentical $s_j \in N$.

(3) Assume that there is no $s_n \in N$ such that s_nPs_j for every other nonidentical $s_j \in N$. Consider the element s_n of N that the element s_n' is mapped with. By assumption, there must be some $s_j \in N$ such that s_jPs_n. However, by statement 1, there can be no $s_j' \in N'$ such that $s_j'P's_n'$. Because f: $S' \to S$ preserves the relation P' on S', there can be no $s_j' \in N'$ such that s_j' is mapped with s_j by f. Thus, f: $S' \to S$ leaves a gap on L between M and s_n.

(4) On the other hand, assume that there is some $s_n \in N$ such that s_nPs_j for all other nonidentical $s_j \in N$. Under this assumption the converse of statement 3 holds and f: $S' \to S$ does not leave a gap on L between M and s_n.

(5) Statement 3 may be generalized to all ordinal partitions of the system [S, P] in which the "upper" subset has no "first" element. Statement 4 may be generalized to all ordinal partitions of the system [S, P] in which the "upper" subset has a "first" element.

Developmental representation of Model III. Suppose we again interpret S' as D and P' as T. By assumption, conditions 1 through 5 for continuity hold under these interpretations. By our ammendment of condition 6, the system [D, T] is not complete. Under these interpretations, the preceding informal proof establishes that, if we were to map each and every behavioral event in D with one and only one point along L in such a manner that "precedes in time" is preserved by "to the left of," this mapping would leave a gap on L between some pairs of nonidentical points (s_i, s_j) such that $s_i P s_j$. The pairs of points between which gaps are left are those for which: (1) s_j is a member of the "upper" subset of some ordinal partition of that has no "first" element; (2) s_j is mapped with the "first" element of the "upper" subset of some ordinal partition of [S', P']; (3) s_i is a member of the "lower" subset of the same ordinal partition of [S, P] and that "lower" subset has no "last" element. On the other hand, when s_j is the "first" element of the "upper" subset of some ordinal partition of [S, P] whose "lower" subset, of which s_i is a member, has no "last" element, the mapping of [S', P'] onto L leaves no gap between s_i and s_j.

Numerical representation of Model III. If S' is interpreted as the set of all rational numbers and P' is interpreted as the relation "less than," then the system [P', S'] is a model of the rational numbers. In other words, the series of rational numbers is the appropriate interpretation of Model III. A rational number again is taken to be any number that can be expressed in terms of a quotient of the form $\underline{a}/\underline{n}$, where $\underline{n}$ is any number from the set 1, 2, 3, ... and $\underline{a}$ is any number from the set 0, ±1, ±2, ±3, ... As was the case for the numerical representations of Models I and II, the properties of the series of rational numbers are well-known. These properties are simple ordering, complete unboundedness, density, and incompleteness [e.g., cf. BEAUMONT and PIERCE, 1963; FEFERMAN, 1964; NIVEN, 1961]. Hence, if each and every rational number is mapped with one and only one point along L in such a manner that "less than" is preserved by "to the left of," the mapping will leave gaps between all pairs of nonidentical points ($\underline{s}_i$, $\underline{s}_j$) such that $\underline{s}_i P \underline{s}_j$ for which $\underline{s}_j$ is a number of the "upper" subset of some ordinal partition of [S, P] that has no "first" element and $\underline{s}_i$ is a member of the "lower" subset of that same ordinal partition where the "lower" subset has no "last" element.

General Discussion

According to the present analysis, then, it is the behavioral versions of the properties of density and completeness which differentiate the three statements about behavioral development with which we began. The statement that behavioral development is continuous is reducible to the contention that, like L, the system [D, T] is both dense and complete. The statement that behavioral development is discontinuous is reducible to the contention that, like the integers, the system [D, T] is neither dense nor complete. The statement that behavioral development is partially discontinuous (and, therefore, partially continuous) is reducible to the contention that, like the rationals, the system [D, T] is dense but not complete.

It is obvious that these three results apply to narrowly defined content areas of behavioral development just as readily as they apply to behavioral development in general. For purposes of maximum generality, we assigned the elements of D their broadest possible empirical interpretations in our preliminary definition. It will be recalled, however, that each of our three formal models were formulated with reference to an abstract system [S', P'] which was assigned no particular empirical interpretation. Therefore, if one wishes to restrict the continuity question to some narrowly defined content area such as social development, perceptual development, language development, etc., then one has only to reinterpret the elements of D in an appropriate manner. In view of the fact that the continuity question traditionally has been posed for restricted behavioral content areas such as those just noted rather than for behavioral development as a whole, the generalizability of the present analysis to any and all such content areas is a point of considerable pragmatic significance.

Continuity and Partial Discontinuity

Although the primary goal of this analysis has been to provide a purely formal explication of the three statements with which we began rather than to pass judgment on their respective validity, it should be noted that the three statements (and, therefore, their corresponding formal models) are not of equal importance from the standpoint of existing [illegible]. The continuity model certainly would have a large [illegible] many developmental psychologists. Also, [illegible] behavioral development is at least partially discontinuous.

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Continuity and Partial Discontinuity

Although the primary aim of this analysis has been to provide a purely formal explication of the three statements with which we began rather than to pass judgment on their respective validity, it should be observed that the three statements (and, therefore, their corresponding formal models) are not of equal importance from the standpoint of existing developmental theories. The continuity model certainly would have a large group of adherents among contemporary developmental psychologists. Also, the statement that behavioral development is at least partially discontinuous has been forcefully advocated by developmental theorists who subscribe to the so-called "organismic" viewpoint [e.g., INHELDER, 1962; PIAGET, 1956, 1960; WERNER, 1948, 1957]. However, the statement that behavioral development is completely discontinuous currently has no identifiable group of advocates. Hence, although all three of our models certainly are of formal interest, only Model I and Model II could be termed "important" by the criterion of existing developmental theories.

A noteworthy point follows from the fact that Model II seems to be of only academic significance at present: For practical purposes, the continuity question (regardless of whether it is posed for behavioral development as a whole or for some narrowly defined content area) reduces to a difference of opinion over the completeness property. This point is noteworthy because I think it is somewhat counterintuitive. Intuitively, it is the fact that one can always find an element between any two given elements (density) that seems to capture the essence of continuity. The truth of this contention is suggested by the fact that the density property of continua was discovered some 2500 years before Dedekind discovered the completeness property. Given presently available developmental theories, however, the density property is not an important issue at all. It is by virtue of the completeness property that continuous and partially discontinuous statements about behavioral development are to be distinguished.

In view of the preceding claims, some brief remarks should be made about how, in terms of our formal models, one goes about postulating either continuity or partial discontinuity in some specific content area of behavioral development. There is only one way to postulate continuity:

If we stipulate that the domain D contains as many elements as there are points on L, then the system [D, T] satisfies Model I. Or, more precisely, Model I is satisfied by [D, T] if the cardinal numbers of D and L are identical. In contrast, partial discontinuity can be postulated in a variety of more or less equivalent ways. Generally speaking, if the cardinal number of D is the same as the cardinal number of the set of all rationals, then [D, T] satisfies Model III. Two overlapping versions of partial discontinuity have been posited in the writings of organismically-oriented developmental theorists: (1) some behavioral content areas are characterized by completely continuous development (typical examples: perception, fine motor skills) while other content areas are characterized by at least some discontinuous development (typical examples: cognition, language); (2) in some or all behavioral content areas, "phases of continuity alternate with phases of discontinuity [INHELDER, 1962, p. 24]." These two forms of partial discontinuity may be said to overlap because the second constitutes a definition of the notion of "at least some discontinuous development" mentioned in the first. Only the second of the two forms seems to characterize WERNER'S developmental theory [cf. WERNER, 1948, 1957]. However, both of these forms of partial discontinuity are mentioned in PIAGET'S theory [cf. PIAGET, 1956, 1960, 1967; PIAGET and INHELDER, 1969; INHELDER, 1962].

Empricial Implications

Because the point of departure for the present analysis was an extant theoretical problem, this paper has been concerned primarily with conceptual issues. It therefore would seem apparent that increased theoretical precision is the principal benefit to be derived from the analysis. It turns out, however, that certain aspects of the analysis also bear on the methodological problem of how the continuity question is to be researched. Explicitly, certain aspects of the analysis suggest methodological innovations which might result in a more perspicuous empirical formulation of the continuity question than those which have been common in the developmental literature to date. In closing, I should like to discuss some of these innovations briefly.

As EMMERICH [1964, 1966] and others [e.g., AUSUBEL, 1954] have pointed out, developmental research on the continuity question traditionally has been predicated on one or the other of two empirical formulations of the question: "The first considers behavioral continuity over time and asks if needs, acts, cognitive operations, etc. are essentially the same at various periods of development .... The other approach defines the continuity issue in terms of individual stability. Here, the essential question is whether distinctiveness of the individual relative to others is maintained throughout development ... [EMMERICH, 1964, pp. 311-312]." In line with EMMERICH'S description, these contrasting approaches may be termed the nomothetic and idiographic formulations of the continuity question, respectively. Unfortunately, both formulations share a well-known logical weakness of which most developmental psychologists are only too painfully aware, namely, "How much is enough?" The empirical findings adduced in the context of either formulation are principally correlational. Neither formulation tends to produce either correlations that approach the square root of the reliability of one's measures (complete continuity) or in correlations that approach zero (partial discontinuity). Moderate correlations tend to be the rule. Hence, questions of the form, "How much stability equals continuity?" and "How much instability equals partial discontinuity?" invariably crop up. Obviously, answers to such questions will be fundamentally arbitrary.

The present analysis suggests an empirical formulation of the continuity question which is both more precise and logically less suspect than either of the preceding formulations. It will be recalled that each of the three models developed above has a numerical representation that corresponds to one of three well-known number systems. In view of the fact that the axioms of the relevant number system and the formal properties of behavioral development are isomorphic in each of the three models, it follows that <u>all and every theorem of the relevant number system is also a theorem of behavioral development</u> in each instance. That is, every statement that is true about the relevant number system for the model also must hold for behavioral development. (In mathematics, systems with isomorphic axiom sets are termed "categorical." One important property of categorical systems is that their theorems are the same [e.g., cf. STABLER, 1953].) Of course, it is necessary to assign the axioms of each model their developmental interpretations before the number theorems become formal theorems of development. In principle, then, the empirical validity of our three statements about behavioral development may be differentially assessed for distinct behavioral content areas as follows. First, derive the following theorem sets: integer theorems, rational number theorems, real number theorems. Second, choose the following subset of theorems from each set: those theorems which serve to differentiate that set from the other two theorem sets. Third, assign the axioms of each number system their developmental interpretations. Fourth and finally, conduct empirical studies of each theorem subset chosen in step two. The model that is isomorphic with the number system whose theorem subset receives the most consistent support from the data then will be taken to be the appropriate model for development in the behavioral content area under investigation.

The procedure just outlined is far less complicated than it sounds. Number theory is one of the oldest and most extensively researched branches of pure mathematics. The important theorems on which the integers, rational numbers, and real numbers differ are well-known and they may be found in all standard works on the common number systems or the foundations of algebra [e.g., FEFERMAN, 1964; BEAUMONT and PIERCE, 1963]. Hence, the most difficult part of the procedure already has been completed. The first and second steps require nothing more than a careful search of the number theory literature. Because the third step is simply interpretational, it is only the final step that remains to be executed whenever one wishes to apply this procedure to any given behavioral content area.

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Footnote

[1]It is possible to state the completeness property in the opposite manner--i.e., whenever the "upper" segment of an ordinal partition of a continuous system contains no "first" element, the "lower" segment of that same ordinal partition may or may not contain a "last" element. These two formulations of the completeness property are equivalent. As a matter of convention, however, mathematicians typically use the formulation appearing above [LUCHINS and LUCHINS, 1965].